## ON A PROBLEM OF TRACKING

### (OB ODNOI ZADACHE PRESLEDOVANIIA)

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In the note are derived certain relationships characterizing the rendezvous problem of two linearly controlled motions.

**1.** Let us consider two motions: a tracked motion $z(t)$ described by the equation

$$\frac{dz}{dt} = C(t) z + g(t) + d(t) v \qquad (1.1)$$

and a tracking motion $y(t)$ described by the equation

$$\frac{dy}{dt} = A(t) y + f(t) + b(t) u \qquad (1.2)$$

Here $y$ and $z$ are $n$-dimensional vectors of phase coordinates $y_i$ and $z_i$ of the tracked objects; $A(t)$ and $C(t)$ are $n \times n$ matrices; $b(t)$ and $d(t)$ are $n$ vectors of system parameters; $f(t)$ and $g(t)$ are $n$ vectors of external disturbances; $u$ and $v$ are scalar control variables. The functions $A(t)$, $C(t)$, $b(t)$, $d(t)$, $f(t)$ and $g(t)$ are defined for $t \geqslant 0$.

The optimal time tracking problem is considered in [1, p.250, 2]. An evaluation of the tracking time for one such problem is given in [3]. Below are given certain relationships which characterize the rendezvous problem of the motions (1.1) and (1.2), subject to typical limits on the control actions $u(t)$ and $v(t)$.

We will formulate the time optimal rendezvous problem of the motions $y(t)$, (1.2) and $z(t)$, (1.1).

Let us consider three types of restrictions on the control actions [4, p.625].

1) Restriction on the action maximum

$$|u(t)| \leqslant N, \qquad |v(t)| \leqslant M \qquad (t \geqslant t_0) \tag{1.3}$$

2) Restriction on the action "energy"

$$\int_{t_0}^{\infty} u^2(t)\,dt \leqslant N^2, \qquad \int_{t_0}^{\infty} v^2(t)\,dt \leqslant M^2 \tag{1.4}$$

3) Restriction on the action impulse

$$\int_{t_0}^{\infty} |d\eta(t)| \leqslant N, \qquad \int_{t_0}^{\infty} |d\zeta(t)| \leqslant M \qquad \begin{pmatrix} d\eta(t) = u(t)\,dt \\ d\zeta(t) = v(t)\,dt \end{pmatrix} \tag{1.5}$$

Here $t_0$ is the beginning of the control process, and the integrals (1.5) have the sense of full variations [5, p.184] of the function $\eta(t)$ and $\zeta(t)$ in the interval $[t_0, \infty]$. The class of permissible functions $u(t)$ and $v(t)$ ($\eta(t)$ and $\zeta(t)$ in the case of (3)) are determined in each case by the character of the restrictions (1.3) to (1.5). In case of (1) the functions $u(t)$ and $v(t)$ are measureable [5, p.20], in case of (2) the functions $u(t)$ and $v(t)$ are determined by the integrable square for $t_0 \leqslant t < \infty$ [5, p.28], in case of (3) the functions $\eta(t)$ and $\zeta(t)$ are determined by the bounded variation on the interval $(t_0, \infty)$ [5, p.184]. In addition, we will restrict the sets of permissible control functions $u(t)$ and $v(t)$ and the corresponding motions $y(t)$ and $z(t)$ also by the following condition: in the case of (1) and (2), the functions $u(t)$ and $v(t)$ must be continuous on the right, and in the case (3) the motions $y(t)$ and $z(t)$ must also be continuous on the right.

We will say that the control functions $u(t)$ and $v(t)$ are bounded by the restrictions $(i)$, $(j)$, meaning the restrictions (1), (2) and (3), and will write $u \in (i)$, $v \in (j)$ if the function $u(t)$ is bounded by the restriction $(i)$ ($i$ = (1), (2), (3)), and the function $v(t)$ is bounded by the restriction $(j)$ ($j$ = (1), (2), (3)).

For the given initial conditions $t = t_0 \geqslant 0$, $y(t_0) = y^\circ$ and $z(t_0) = z^\circ$, and for the selected control functions $u(t) \in (i)$, $v(t) \in (j)$ $(t \geqslant t_0)$ the time $t = t^*$ is called the instant of interception [1,2], if $y(t^*) = z(t^*)$ $(t^* \geqslant t_0)$. If for some $t = t^*$ the motions $y(t)$ and $z(t)$ meet for the first time, i.e. if $y(t^*) = z(t^*)$ but $y(t) \neq z(t)$ for $t_0 \leqslant t < t^*$, then this instant $t^*$ will be called the first instant of interception and denoted as $t_1^*[t_0, y^\circ, z^\circ, u, v, i, j]$. With the introduced restrictions on the class of permissible functions $u(t)$ and $v(t)$ there will exist a first instant of interception for any two fixed motions $y(t)$ and $z(t)$ $(t \geqslant t_0)$ possessing at least one instant of interception $t = t^*$.

With the restrictions (1) to (3), the problem of [1,2] transforms into the following problem.

*Problem* $[(i), (j)]$. For given $t_0$, $y^0$ and $z^0$, find optimal control functions $u^0(t) \in (i)$ and $v^0(t) \in (j)$, for which

$$t_1{}^* [t_0, y^\circ, z^\circ, u^\circ, v^\circ, (i)\,(j)] = \max_v \min_u t_1^* [t_0, y^\circ, z^\circ, u, v, (i), (j)] \qquad (1.6)$$

This formulation of the problem is less natural than the formulation of the minimax control problems in the theory of dynamic programming [6]. In the dynamic programming problems* the optimal control functions $u$ and $v$ are sought usually in the form of the functions $u = U[t, y, z]$ and $v = V[t, y, z]$ and the control actions are determined consequently, at each instant of time $t$ of the process from the realized quantities $y(t)$ and $z(t)$, i.e.

$$u(t) = U[t, y(t), z(t)], \qquad v = V[t, y(t), z(t)]$$

At the same time it is assumed in the problems of dynamic programming that the strategy of $z(t)$ ($v(\tau)$ for $\tau \geqslant t$) is not known at time $t$ in the control element $y(t)$, while the strategy of $y(t)$ ($u(\tau)$ for $\tau \geqslant t$) is not known in the control element $z(t)$. Such an assumption corresponds to a typical situation in the theory of games [7]. In the problem $[(i), (j)]$ it is assumed, however, that the control function $v(t)$ for all $t \geqslant t_0$ is communicated ahead into the element producing the control function $u(t)$. In spite of this shortcoming the problem $[(i), (j)]$ is apparently important for its own interest as well as an auxiliary link in other problems.

2. The necessary conditions which must be satisfied by the optimal control functions $u^0(t)$ and $v^0(t)$ in the problem $[(i), (j)]$ for $i = 1$ and $j = 1$ are in [1,2]. Such conditions can be derived also for other problems $[(i), (j)]$. Here, however, such conditions are of little use for an effective solution of the problem. The bounds of the considered problems are therefore of interest as well as the consideration of other problems of combining the motions $y(t)$ and $z(t)$. The latter is important also because in the wide class of cases the problem $(i, j)$ can have no solution (see below). Sufficiently accurate evaluation of the quantity $t_1{}^*[t_0, y^0, z^0, u^0, v^0, (i), (j)]$, even if it exists, is apparently very difficult. Therefore, we will consider in the following another quantity

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* Tracking problems, optimal with respect to the instant of rendezvous, where investigated by Iu.M. Repin from the position of the theory of dynamic programming at the seminar on differential equations in Sverdlovsk in 1957. Some problems of tracking of analogous type were considered in the author's report at the IVth All-Union mathematics conference (conference program p. 68).

$T$ which bounds the quantity $t_1^*$ from above.

*Definition 2.1.* Let there be given initial conditions $t_0$, $y^\circ$, $z^\circ$ and the restrictions $u \in (i)$, $v \in (j)$. The time $t = T \geqslant t_0$ will be called the instant of absorbing of the $z(t)$ (1.1) process by the $y(t)$ process (1.2), if for any control function $v(t) \in (j)$ there is a $u(t) \in (i)$ for this $v(t)$ such that the control functions $v(t)$ and $u(t)$ realize the interception of motions of $y(t)$ and $z(t)$ at time $t = T$. If there exists the smallest $T$, it will be called the first instant of absorption of the process $z(t)$ by the process $y(t)$. The first instant of absorption will be denoted by $T_1[t_0, y^\circ, z^\circ, (i), (j)]$.

It is this type of quantities $T$ that are evaluated in [3] by the disturbance accumulation theory for the problem [(1), (1)] in the case of the second order systems (1.1) and (1.2).

Let us investigate the relation between the quantities $T_1$ and $t_1^*$. If there exists a certain instant of absorption $T$ then, apparently, the inequality

$$\sup_v \inf_u t_1^* \, [t_0, y^\circ, z^\circ, u, v, (i), (j)] \leqslant T \tag{2.1}$$

is valid.

Consequently, if there exists an optimal control function $u^\circ$ and $v^\circ$ for which

$$t_1^* \, [t_0, y^\circ, z^\circ, u^\circ, v^\circ, (i), (j)] = \max_v \min_u t_1^* \, [t_0, y^\circ, z^\circ, u, v, (i), (j)] \tag{2.2}$$

and if there exists the first instant of absorption $T_1$, then

$$t_1^* \, [t_0, y^\circ, z^\circ, u^\circ, v^\circ, (i), (j)] \leqslant T_1 \, [t_0, y^\circ z^\circ, (i), (j)] \tag{2.3}$$

If there exists at least one $T$, then the existence of $T_1$ is provable for sufficiently general assumptions. (See p. 370 below). The existence of max min $t_1^*$ even for the condition of bounded sup inf $t_1^*$ is, on the contrary, not provable in even such a general case. This represents the essential difference of the problem $[(i), (j)]$ from the analogous problems on the optimal with respect to the speed of response controls [4] where from the existence of at least one permissible control function $u(t)$ bringing the motion $x(t)$ into a given point, there follows the existence of the optimal control function $u^\circ(t)$. We also note that, generally speaking, the existence of even one instant of absorption $T$, does not follow from the existence of the optimal control functions $u^\circ(t)$, $v^\circ(t)$.

The following material of the article investigates the rendezvous of the motions $z(t)$ and $y(t)$ at the instant of absorption $T$ of the process $z(t)$ by the process $y(t)$.

**3.** Let us derive the bounds characterizing the instant of absorption $T$ of the process $z(t)$ by $y(t)$. Let $\varphi(t)$ be a function defined for $t_0 \leqslant t \leqslant \tau$. Let us denote

$$\|\varphi(t), t_0, \tau, (1)\| = \int_{t_0}^{\tau} |\varphi(t)|\, dt \tag{3.1}$$

$$\|\varphi(t), t_0, \tau, (1)\|^* = vrai \max [\,|\varphi(t)|, t_0 \leqslant t \leqslant \tau] \tag{3.2}$$

$$\|\varphi(t), t_0, \tau, (2)\| = \|\varphi(t), t_0, \tau, (2)\|^* = \Big[\int_{t_0}^{\tau} \varphi^2(t)\, dt\Big]^{1/2} \tag{3.3}$$

$$\|\varphi(t), t_0, \tau, (3)\| = \max [\,|\varphi(t)|, t_0 \leqslant t \leqslant \tau] \tag{3.4}$$

$$\|\varphi(t), t_0, \tau, (3)\|^* = \int_{t_0}^{\tau} |d\varphi(t)| \tag{3.5}$$

In each of the conditions (3.1) to (3.5) the class of functions $\varphi(t)$ is assumed such that the corresponding norm $\| \varphi(t), t_0, \tau, (i) \|$ or $\| \varphi(t), t_0, \tau, (i) \|^*$ is meaningful.

It is known that for each $(i)$ $(i = (1), (2), (3))$ the quantities $\| \varphi(t), t_0, \tau, (i) \|$ and $\| \varphi(t), t_0, \tau, (i) \|^*$ define a metric in standard functional spaces ($L$, $L^2$ and $C$ respectively) and the norms of their linear functionals [5, p.165]. The restrictions (1.3) to (1.5) in the notation of (3.2), (3.3), (3.5) are of the form

$$\|u(t), t_0, \infty, (i)\|^* \leqslant N, \qquad \|v(t), t_0, \infty, (i)\|^* \leqslant M \quad (i = 1,2) \tag{3.6}$$

$$\|\eta(t), t_0, \infty, (i)\|^* \leqslant N, \qquad \|\zeta(t), t_0, \infty, (i)\|^* \leqslant M \quad (i = 3) \tag{3.7}$$

We will use the symbols $F^{(1)}[t_0, t]$ and $F^{(2)}[t_0, t]$, respectively for the fundamental matrix of solutions of equations

$$dz\,/\,dt = C(t)\, z, \qquad dy\,/\,dt = A(t)\, y$$

and the symbols $(F^{(k)}[t_0, t])^{-1}$ for the inverse matrices of $F^{(k)}$. It is assumed that $F^{(k)}[t_0, t_0] = E$ the unit matrix. By $[q]_k$ will be denoted the $k$th component of the vector $q$.

Let us evaluate the regions $S^{(1)}[t_0, T, z^0, (j)]$ consisting of those points $z$ into which one can reduce the motion $z(t)$ at time $t = T$ originating from the point $z(t_0) = z^0$ by a choice of the control function $v(t) \in (j)$. We will write down the solution of equation (1.1) according to the Cauchy formula [8, p.172] as

$$z(T) = F^{(1)}[t_0, T]\, z^\circ + \int_{t_0}^{T} F^{(1)}[t_0, T]\, (F^{(1)}[t_0, t])^{-1} \{g(t) + d(t)\, v(t)\}\, dt \quad (3.8)$$

or in the coordinates we obtain the equalities

$$c_k[t_0, T] = \int_{t_0}^{T} h_k^{(1)}[t_0, T, t]\, v(t)\, dt \qquad (k = 1, \ldots, n) \quad (3.9)$$

where

$$c_k[t_0, T] = z_k(T) - \left[F^{(1)}[t_0, T]\, z^\circ + \int_{t_0}^{T} F^{(1)}[t_0, T]\, (F^{(1)}[t_0, t])^{-1}\, g(t)\, dt\right]_k \quad (3.10)$$

$$h_k^{(1)}[t_0, T, t] = [F^{(1)}[t_0, T]\, (F^{(1)}[t_0, t])^{-1}\, d(t)]_k \quad (3.11)$$

The set of values $c = \{c_k\} \neq 0$ for which the equations (3.9) are solvable with respect to the functions $v(t) \in (j)$ is defined by the condition [4]

$$\Phi^{(1)}[t_0, T, c, (j)] \geqslant \frac{1}{M} \quad (3.12)$$

where

$$\Phi^{(1)}[t_0, T, c, (j)] = \min_\lambda \|\lambda \times h^{(1)}[t_0, T, t], t_0, T, (j)\| \quad \text{for } \lambda \times c = 1 \quad (3.13)$$

The symbol $\lambda \times q$ denotes here the scalar product of the vectors $\lambda$ and $q$. Also, the equations (3.9) are solvable for $c_k = 0$ $(k = 1, \ldots, n)$.

Thus, the region $S^{(1)}$ consists of such points $z$ for which either $c = 0$ or $c \neq 0$ and the condition (3.12) is fulfilled, with the coordinates $z_k$ and $c_k$ being connected by the relation (3.10) for $z_k = z_k(T)$. This set $S^{(1)}$ will be bounded, closed and convex [9].

One can also conclude from the condition (3.12) that the boundary of the region $S^{(1)}[t_0, T, z^\circ, (j)]$ varies continuously for continuous variation in $t_0$, $T$ and $z^\circ$.

If $T$ is the instant of absorption, then for any control function $v^*(t) \in (j)$ one can indicate a function $u^*(t) \in (i)$ such that for the corresponding motions the equality $y^*(T) = z^*(T)$ is valid.

Consequently, the region $S^{(1)}$ must be within the region $S^{(2)}[t_0, T, y^\circ, (i)]$ each point of which is accessible by the motion $y(t)$ at $t = T$ originating from the point $y(t_0) = y^\circ$ for the control functions $u(t) \in (i)$.

The region $S^{(2)}$ is defined analogously to the region $S^{(1)}$ by the condition

$$\text{either } a = 0 \quad \text{or} \quad \Phi^{(2)}[t_0, T, a, (i)] \geqslant \frac{1}{N} \qquad (a \neq 0) \tag{3.14}$$

where

$$\Phi^{(2)}[t_0, T, a, (i)] = \min_\lambda \| \lambda \times h^{(2)}[t_0, T, t], t_0, T, (i) \| \quad \text{for } \lambda \times a = 1 \tag{3.15}$$

moreover

$$a_k[t_0, T] = z_k - [F^{(2)}[t_0, T]\, y^\circ + \int_{t_0}^{T} F^{(2)}[t_0, T] \times (F^{(2)}[t_0, t])^{-1} f(t)\, dt]_k \tag{3.16}$$

$$h_k^{(2)}[t_0, T, t] = [F^{(2)}[t_0, T]\, [F^{(2)}[t_0, t])^{-1} b(t)]_k \tag{3.17}$$

Consequently, $t = T$ is the absorption instant if and only if, the condition

$$\min_a \Phi^{(2)}[t_0, T, a, (i)] \geqslant \frac{1}{N} \qquad (a \neq 0) \tag{3.18}$$

is fulfilled (the case when $S^{(1)} = S^{(2)}$ is the point, is not interesting) for

$$\Phi^{(1)}[t_0, T, c, (j)] \geqslant \frac{1}{M} \qquad (c \neq 0) \quad (\text{and for } c = 0)$$

where the vectors $a$ and $c$ are connected by the relation

$$a = c + F^{(1)}[t_0, T]\, z^\circ + \int_{t_0}^{T} F^{(1)}[t_0, T]\, (F^{(1)}[t_0, t])^{-1} g(t)\, dt -$$

$$- F^{(2)}[t_0, T]\, y^\circ - \int_{t_0}^{T} F^{(2)}[t_0, T]\, (F^{(2)}[t_0, t])^{-1} f(t)\, dt \tag{3.19}$$

In view of the convexity of the regions $S^{(1)}$ and $S^{(2)}$, it is sufficient to seek the minimum of (3.18) under the condition

$$\Phi^{(1)}[t_0, T, c, (j)] = \frac{1}{M} \qquad (c \neq 0) \quad (\text{and for } c = 0) \tag{3.20}$$

The solution of problem (3.18) is in the general case quite difficult.

We note in conclusion that the quantities $\varphi^{(i)}(t) = \lambda \times h^{(i)}[t_0, T, t]$ have also the following meaning here: $\varphi^{(1)}(t)$ is the scalar [1] product of the vector $d(t)$ by the vector-solution $\psi^{(1)}(t)$ of the equation $d\psi/dt = - C^*(t)\psi$ satisfying the initial conditions $\psi^{(1)}(T) = \lambda$, while $\varphi^{(2)}(t)$ is the scalar product of the vector $b(t)$ by the vector-solution $\psi^{(2)}(t)$ of the equation $d\psi/dt = - A^*(t)\psi$, satisfying the condition $\psi^{(2)}(T) = \lambda$. The star (*) denotes transposition.

4. Let us consider the question of determining the first absorption instant $T_1$ of process $z(t)$ by the process $y(t)$. We will assume that the motion $y(t)$ will be controlled [10, 11, p.222] on each segment $t_0 \leqslant t \leqslant \tau$ ($\tau > t_0$). For this it is sufficient that the vectors

$$L_1^{(2)}(t) = b(t), \qquad L_{k+1}^{(2)}(t) = \frac{dL_k^{(2)}}{dt} - A(t) L_k^{(2)}(t) \qquad (k = 1, \ldots, n-1) \tag{4.1}$$

be linearly independent for $t = t_0$.

In this case the first instant of absorption $T_1$ is defined as the smallest root of the equation (for $i \neq 3$, $j \neq 3$)

$$\min \Phi^{(2)}[t_0, T, a, (i)] = \frac{1}{N}$$
$$\text{for} \quad \Phi^{(1)}[t_0, T, c, (j)] = \frac{1}{M} \quad (c \neq 0) \quad (\text{and for } c = 0) \tag{4.2}$$

which exists if there exists at least one instant of absorption $T$.

Indeed, $T_1$ is the smallest number among $T$ satisfying the condition

$$\min_a \Phi^{(2)}[t_0, T, a, (i)] \geqslant \frac{1}{N} \ \text{for} \ \Phi^{(1)}[t_0, T, c, (j)] = \frac{1}{M} \quad (c \neq 0)(\text{and for } c = 0) \tag{4.3}$$

If $t = T$ is some instant of absorption then for this $T$ the condition (4.3) is fulfilled. On the other hand, for $y^\circ \neq z^\circ$ and for $T \to t_0$ the set (4.3) tends to a point $c = 0$ and, consequently, the quantity $\min_a \Phi^{(2)}[t_0, T, a, (i)]$ tends to zero since the segment $(t_0, T)$ tends to a point while the vector $a = z^\circ - y^\circ + c + 0[T - t_0]$ remains finite and non-zero. Now, in order to prove the existence of the smallest root of equation (4.2) it is sufficient to note that the quantity $\min_a \Phi^{(2)}$ for $\Phi^{(1)} = 1/M$ varies continuously with $T$. The latter condition is fulfilled since under the conditions of control the quantity $\Phi^{(2)}$ depends continuously on $T$ and $a$, while the set (4.3), as noted previously, deforms continuously with $T$, and the vectors $c$ and $a$ are connected by a relationship continuous in $T$.

If the condition for controllability of $y(t)$ is not fulfilled, then at the first absorption instant $T_1$, the strict inequality (4.3) rather than (4.2) may be fulfilled, since $\Phi^{(2)}$ can be discontinuous in $a$.

Thus, the instant $T_1$ can in a number of cases be determined from the equation (4.1). The practical solution of this equation is quite difficult. One can utilize the method of introducing the parameter $\vartheta$, into the problem as it is described in [4] for the problem of optimal control with respect to the speed of response. However, such an approach can be complicated by bifurcation points, the absence of which cannot be established in advance.

**5.** Let us consider the case of restrictions [(2), (2)] when the computation of the quantity $\min_a \Phi^{(2)}$ for $\Phi^{(1)} = 1/M$ is least complicated. We will assume, for simplicity, that not only the motion $y(t)$ but also the motion $z(t)$ is controlled. For this it is sufficient that along with the vectors $L_k^{(2)}$ (4.1), the vectors

$$L_k^{(1)}(t_0) \,/\, L_1^{(1)}(t) = d(t), \quad L_{k+1}^{(1)}(t) = dL_k^{(1)}/dt - C(t) L_k^{(1)}(t) \qquad (k = 1, \ldots, n)$$

also be linearly independent.

In place of $\Phi^{(1)}$ and $\Phi^{(2)}$ it is convenient here to consider the quantities

$$\alpha^{(i)}[t_0, T, q] = (\Phi^{(i)}[t_0, T, q, (2)])^{-2} \tag{5.1}$$
$$(i = 1,2;\ q = a \text{ for } i = 2,\ q = c \text{ for } i = 1)$$

Then the conditions (4.3), which are fulfilled at each instant of time $T$, are transformed into conditions

$$\max_a \alpha^{(2)}[t_0, T, a] \leqslant N^2 \quad \text{for} \quad \alpha^{(1)}[t_0, T, c] = M^2 \tag{5.2}$$

Quantities $\alpha^{(i)}$ are quadratic forms of the quantities $c_j$ and $a_j$, respectively, for $i = 1$ and $i = 2$. Since the vectors $\{a_j\}$ and $\{c_j\}$ are connected with the vector $\{z_j\}$ by the linear relations (3.10) and (3.16), the problem (5.2) is reduced to the problem

$$\max_a \alpha^{(2)} = \max_z \Big\{ \sum_{i,j=1}^{n} \beta_{ij}^{(2)}(t_0, T)\, z_i z_j + \sum_{i=1}^{n} \beta_i^{(2)}(t_0, T)\, z_i + \beta^{(2)}(t_0, T) \Big\} \tag{5.3}$$

for

$$\alpha^{(1)} = \sum_{i,j=1}^{n} \beta_{ij}^{(1)}(t_0, T)\, z_i z_j + \sum_{i=1}^{n} \beta_i^{(1)}(t_0, T)\, z_i + \beta^{(1)}(t_0, T) = M^2 \tag{5.4}$$

where $\Sigma \beta_{ij}^{(k)} z_i z_j$ $(k = 1, 2)$ are positive definite forms. The coefficient $\beta_{ij}^{(k)}$, $\beta_i^{(k)}$ and $\beta^{(k)}$ are computed as follows:

$$\alpha^{(k)} = \sum_{i,j=1}^{n} \beta_{ij}^{(k)} q_i q_j = \int_{t_0}^{T} \Big( \sum_{i=1}^{n} \rho_i^{(k)} h_i^{(k)}[t_0, T, t] \Big)^2 dt \qquad \begin{pmatrix} q = a \text{ for } k = 2 \\ q = c \text{ for } k = 1 \end{pmatrix} \tag{5.5}$$

where $\{\rho_i^{(k)}\}$ is the solution of the system of equations [10]

$$\sum_{j=1}^{n} (h_i^{(k)} \times h_j^{(k)})\, \rho_j^{(k)} = (h_i^{(k)} \times q) \tag{5.6}$$

$$(h_i^{(k)} \times h_j^{(k)}) = \int_{t_0}^{T} h_i^{(k)} h_j^{(k)} dt, \qquad (q_i \times h_i^{(k)}) = q_i \int_{t_0}^{T} h_i^{(k)} dt \tag{5.7}$$

The forms (5.3) and (5.4) are obtained after the substitution of expressions (3.10) and (3.16) for $c$ and $a$ into the form (5.5). The solution of the problem (5.3) and (5.4) is not difficult in principle. However, the solution of this problem is more convenient not in the variables $z_i$ but in certain variables $\lambda_i$ connected with $z_i$ by linear relationships.

Let us consider the expression (5.3). Here $z_i$ are the coordinates of the point $z(T)$ into which can be reduced the motion $z(t)$ at time $t = T$ by the control function $v(t) \in (j)$. Since the point $z(T)$ lies on the surface $\alpha^{(1)} = M^2$, i.e. on the boundary of the region $\alpha^{(1)} \leqslant M^2$ of accessibility, then the control function $v(t)$ on the sector $t_0 \leqslant t \leqslant T$ is optimal in the sense that it minimizes the quantity

$$\varphi = \int_{t_0}^{T} v^2(t)\, dt \tag{5.8}$$

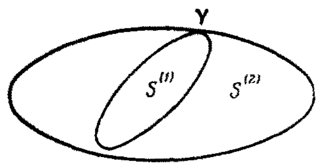


Fig. 1.

for the given boundary conditions $z(t_0)$ and $z(T)$. But the optimal control $v^{\circ}(t)$ is here of the form [10,11]

$$v^{\circ}(t) = \sum_{i=1}^{n} \lambda_i h_i^{(1)} [t_0, T, t] \tag{5.9}$$

Substituting $v = v^{\circ}$ according to (5.9) into (5.8) and (3.8) and substituting the then obtained expression (3.8) into (5.3), we obtain the problem for the maximum of the quadratic function of $\lambda_i$

$$\max_{\lambda} \alpha\, [T, \lambda_1, \ldots, \lambda_n] = ? \tag{5.10}$$

for

$$\varphi\, [\lambda_1, \ldots, \lambda_n, T] = \int_{t_0}^{T} (\lambda_i h_i^{(1)})^2\, dt = M^2 \tag{5.11}$$

**6.** Let us consider the problem of constructing the control function $u$ which would be chosen during the process on the basis of information on the realized values of $z(t)$ and $y(t)$ and which would effect the rendezvous of the motions $y(t)$ and $z(t)$, and moreover $v(\tau)$ for $\tau \geqslant t$

would be unknown in the control element of $y(t)$ at each instant of time $t \geqslant t_0$. The control function $u$ should be constructed such that for the realization $v(t) \in (j)$ there would result the realizations $u(t) \in (i)$.

We will limit ourselves to the simplest case of $i = (2)$, $j = (2)$ for $n = 2$. We assume that the motion $y(t)$ is controlled on each segment $[t_1, t_2] t_0 \leqslant t_1 < t_2$. For this it is sufficient that the vectors $L_k^{(2)}(t)$ $(k = 1, 2)$ (4.1) be linearly independent for all $t \geqslant t_0$ except, perhaps, for separate and isolated values of $t$.

Let there exist at least one instant $t = T > t_0$ of absorption of the process $z(t)$ by $y(t)$ and, consequently, there exists the first instant of absorption $t = T_1(t_0)$ determined from the conditions (4.2). At the instant $t = T_1$, in accordance with the results of Sections 4 and 5, the following situation occurs. The set of points $z$ on the surface $z_1$, $z_2$ into which one can reduce the motion $z(t)$ at the $t = T_1$ subject to the restriction $v(t) \in (2)$ is an ellipse $S^{(1)}(t_0, T_1)$: $\alpha^{(1)} \leqslant M^2$ (Fig. 1) if the motion $z(t)$ is controlled on the segment $t_0 \leqslant t \leqslant T_1$, or this is a set of segments $S^{(1)}(t_0, T_1)$ (Fig. 2) (or even a point) if the motion $z(t)$ is not controlled for $t_0 \leqslant t \leqslant T_1$.

The set $S^{(2)}(t_0, T_1)$ of points $z$ into which one can reduce the motion $y(t)$ at time $t = T_1$ subject to the restriction $u(t) \in (2)$ is an ellipse $\alpha^{(2)} \leqslant N^2$ (Figs. 1, 2). The case when $S^{(i)}(t_0, T_1)$ is a point is not interesting and will not be considered. The set $S^{(1)}(t_0, T_1$ lies within the set $S^{(2)}(t_0, T_1')$ and their boundaries touch at the point $\gamma$ where the maximum of (5.4) results. For all $t = \tau$, $\tau < T_1$ the set $S^{(1)}(t_0, \tau)$ no longer lies entirely within $S^{(2)}(t_0, \tau)$.

Let us assume that the following condition is fulfilled.

*Condition* $Q[t_0, T_1]$. The boundaries of $S^{(2)}(t_0, T_1)$ and $S^{(2)}(t_0, T_1)$ have only one common point $\gamma$, and the curvature of the boundary of $S^{(2)}(t_0, T_1)$ at the point $\gamma$ is smaller than the curvature of the boundary of $S^{(1)}(t_0, T_1)$ at this point.

The latter requirement of the condition $Q[t_0, T_1]$ is always fulfilled if the motion $z(t)$ is uncontrolled for $t_0 \leqslant t \leqslant T_1$.

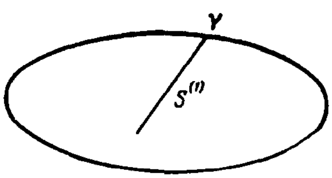


Fig. 2.

Let us select a control function $u(t) \in (2)$ for $t_0 \leqslant t \leqslant T_1$ such that the motion $y(t)$ be reducible by it into a point $\gamma$. This control function $u(t)$ is of the form [10]

$$u(t) = \sum_{i=1}^{2} \rho_i^{(2)} [t_0, T_1] h_i^{(2)} [t_0, T_1, t]$$

where $\{\rho_i^{(k)}\}$ is the solution of the system of equations (5.6) for $T = T_1$, and moreover the quantities $a_i$ and $h_i^{(2)}$ are defined by the equalities (3.16) and (3.17) for $T = T_1$ and $z_i$ equal to the coordinates of the point $\gamma$.

Let on a certain segment $t_0 \leqslant t \leqslant \vartheta(t_0) < T_1$ the control function $u^*(t)$ coincide with the control function $u(t)$ indicated in the previous paragraph, and let the control function $v^*(t)$ coincide on this segment with some control function $v(t) \in (2)$. Let $y^*(\vartheta)$ and $z^*(\vartheta)$ be those points into which are reduced the motions $y(t)$ and $z(t)$ by the control functions $u^*(t)$ and $v^*(t)$ at the time $t = \vartheta(t_0)$. We will consider the control of $y(t)$ and $z(t)$ originating from the points $y^*(\vartheta)$ and $z^*(\vartheta)$ for $t \geqslant \vartheta$, subject to the restrictions

$$\int_{\vartheta}^{\infty} u^2(t)\,dt \leqslant N^2 - \int_{t_0}^{\vartheta} [u^*(t)]^2\,dt = N^2(\vartheta) \tag{6.1}$$

(2) ($\vartheta$)

$$\int_{\vartheta}^{\infty} v^2(t)\,dt \leqslant M^2 - \int_{t_0}^{\vartheta} [v^*(t)]^2\,dt = M^2(\vartheta) \tag{6.2}$$

Let us denote by $S^{(1)}[\vartheta, T_1]$ the region on the surface $z_1$, $z_2$ accessible by the motion $z(t)$ originating from the point $z^*(\vartheta)$ under the control function $v(t)$ subject to the restriction (6.2). Let $S^{(2)}[\vartheta, T_1]$ denote the region accessible by the motion $y(t)$ originating from the point $y^*(\vartheta)$ under the control function $u(t)$ restricted by the condition (6.1). The regions $S^{(1)}[\vartheta, T_1]$ and $S^{(2)}[\vartheta, T_1]$ lie within the regions $S^{(1)}[t_0, T_1]$ and $S^{(2)}[t_0, T_1]$.

The region $S^{(2)}[\vartheta, T_1]$ passes through the point $\gamma$ and, consequently, the boundary of the region $S^{(2)}[\vartheta, T_1]$ touches at this point the boundaries of the region $S^{(2)}[t_0, T_1]$. Under continuous variation of $\vartheta$ the quantities $\beta_{ij}^{(2)}(\vartheta, T_1)$, $\beta_i^{(2)}(\vartheta, T_1)$, $\beta^{(2)}(\vartheta, T)$, which define the form $\alpha^{(2)}$ (5.5), change continuously and therefore the curvature of the ellipse $\alpha^{(2)} = N^2$, which bounds $S^{(2)}$, changes at the point $\gamma$ continuously with variation of $\vartheta$. But then it follows from the condition $Q[t_0, T_1]$ that it is possible to choose such a small segment of time $[t_0, \vartheta^*(t_0)]$, $\vartheta^* > t_0$ that in choosing the control function $u = u^*(t)$ and any control function $v(t) \in (2)$ on this segment, the region $S^{(1)}[\vartheta^*, T_1]$ will be contained inside the region $S^{(2)}[\vartheta^*, T_1]$. Consequently, for the rendezvous problem of the motions $y(t)$ and $z(t)$ subject to the restriction (2) ($\vartheta^*$) originating from the points $y^*(\vartheta^*)$ and $z^*(\vartheta^*)$, the time $t = T_1$ will be the instant of absorption of the process $z(t)$ by the process $y(t)(t \geqslant \vartheta^*)$. Consequently, for this problem there will also exist the first instant of absorption

$$T_1[\vartheta^*, y^*(\vartheta^*), z^*(\vartheta^*), (2)(\vartheta^*), (2)(\vartheta^*)] \leqslant T_1[t_0, y^\circ, z^\circ, (2), (2)]$$

Let us assume that for the instants $\vartheta^*$, $T_1[\vartheta^*, y^*, z^*, (2)(\vartheta^*), (2)(\vartheta^*)] = T_1{}^*$ the condition $Q[\vartheta^*, T_1{}^*]$ (p.374) is again fulfilled. Then, one can utilize the same construction for the segment $[\vartheta^*, T_1{}^*]$ as above for the segment $[t_0, T_1]$. Denoting $\vartheta^* = t_1$, the control function $u^*(t)$ can then be constructed on some segment $t_1 \leqslant t < t_2$. If at the time $t_2$ the condition $Q$ still holds then the construction of $u^*(t)$ can be continued analogously for $t_2 \leqslant t < t_3$ etc. The indicated process can be continued in the direction of increasing $t_k (k = 0, 1, \dots)$ as long as the condition $Q$ holds at each step of the process.

At the same time, there will correspond to the sequence $t_k$ a nonincreasing sequence of the first instants of absorption $T_1[t_k]$. Let us assume that at each step as long as $T_1[t_k] - t_k > \delta > 0$, the curvature of the ellipse-boundary $S^{(1)}[t_k, T_1[t_k]]$ will exceed the curvature of the ellipse-boundary $S^{(2)}[t_k, T_1[t_k]]$ by a constant $\varepsilon(\delta) > 0$.

Then the sequences $t_k$ and $T_1[t_k]$ for $k \to \infty$ will possess a certain common bound $T$ to which the sequence $t_k$ tends from below, and $T[t_k]$ from above. In this case the control function $u(t)$, coinciding with the control function $u^*(t)$ for each $t \in [t_k, t_{k+1}]$, and constructed as was shown above will, apparently, reduce the motion $y(t)$ to the motion $z(t)$ at time $t = T$, no matter how the control function $v(t) \in (2)$ is chosen on each segment $t_k \leqslant t < t_{k+1}$. Here, one can also investigate the limit transition $t_{k+1} \to t_k$, but this is difficult.

Thus, if the condition $Q$ holds at each step $t_k$ of the tracking process, then it is possible to construct the control function $u(t)$ which ensures the rendezvous of the motions $y(t)$ and $z(t)$. If, however, at some step $t_k$ or for $t_k \to t^* < T_1[t^*]$ the condition $Q$ is violated, then the construction of $u(t)$ ensuring the rendezvous of the motions $y(t)$ and $z(t)$, subject to the condition that at time $t$ the choice of $v(t)$ is unknown, may be impossible.

More correctly, it is possible to give an example when there is an instant of absorption of the process $z(t)$ by the process $y(t)$, however, it is not possible to give such a rule for choosing the control function $u(t) = U[t, y, z, N, M]$ continuous at the point $t_0$ at right, which would ensure the rendezvous of the motions $y(t)$, $z(t)$ if at the time $t = t_0$ the function $v(t_0)$ is unknown in the control element of the motion $y(t)$.

*Note 6.1.* In satisfying the conditions of the type $Q[t_0, T_1]$ and for $t_1{}^* = T_1$, one can pass from the solution $u = u^\circ(t)$, $v = v^\circ(t)$ of the problem $[(1), (1)]$ also to the solution $u = U[t, y, z]$, $v = V[t, y, z]$

of the corresponding tracking problem within the aspect of the theory of dynamic programming (see beginning of Section 6) by analogous means considered above for the problem [(2), (2)]. This however, results in difficulties for the required limiting process $t_{k+1} \to t_k$.

*Note 6.2.* The considerations given in the article above are, naturally, generalized for the case when in the equations (1.1) and (1.2) $v$ and $u$ are nonscalars but $r$-vectors, (while $d(t)$ and $b(t)$ are respectively $n \times r$ matrices) and when the rendezvous of $y(t)$ and $z(t)$ is required only on a section of the coordinates. We thus obtain, for example, from purely geometrical considerations, that the instant $t_1{}^* = T_1(t_0 = 0)$ for intersection of the motions of the material points with masses $m^{(1)}$ and $m^{(2)}$

$$\frac{dz_i}{dt} = z_{i+3}, \qquad \frac{dz_{i+3}}{dt} = \frac{1}{m^{(1)}} v_i, \qquad \frac{dy_i}{dt} = y_{i+3}, \qquad \frac{dy_{i+3}}{dt} = \frac{1}{m^{(2)}} u_i \qquad (i = 1, 2, 3)$$

under the condition $y_i(t_1{}^*) = z_i(t_1^*)$ $(i = 1, 2, 3)$ and subject to the restriction

$$u_1^2 + u_2^2 + u_3^2 \leqslant N^2, \qquad v_1^2 + v_2^2 + v_3^2 \leqslant M^2 \qquad \left(\frac{N}{m^{(2)}} > \frac{M}{m^{(1)}} > 0\right)$$

is determined from the equation

$$\sum_{i=1}^{3} [(y_{0i} + T_1 y_{0i+3}) - (z_{0i} + T_1 z_{0i+3})]^2 = \frac{T_1^4}{4}\left(\frac{N}{m^{(2)}} - \frac{M}{m^{(1)}}\right)^2$$

Moreover, the optimal efforts $u^0(u_1, u_2, u_3)$ and $v^0(v_1, v_2, v_3)$ must at each instant of tracking be directed parallel to each other. The latter result was first established by Iu.M. Repin.

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